

## A REMARK ON THE REGULARIZED GAP FUNCTION FOR IQVI

SANGHO KUM\*

ABSTRACT. Aussel et al. [1] introduced the notion of *inverse quasi-variational inequalities* (IQVI) by combining quasi-variational inequalities and inverse variational inequalities. Discussions are made in a finite dimensional Euclidean space. In this note, we develop an infinite dimensional version of IQVI by investigating some basic properties of the regularized gap function of IQVI in a Banach space.

### 1. Introduction

Recently, Aussel et al. [1] introduced the notion of *inverse quasi-variational inequalities* (in short, IQVI) by combining quasi-variational inequalities (QVI) and inverse variational inequalities (IVI) as follows: Given two continuous functions  $F, h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a multifunction  $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with closed convex values  $Sx$  for all  $x \in \mathbb{R}^n$ , the inverse quasi-variational inequality problem, IQVI, is the problem of finding a vector  $\bar{x} \in \mathbb{R}^n$  such that  $h(\bar{x}) \in S\bar{x}$  and

$$\langle F\bar{x}, y - h(\bar{x}) \rangle \geq 0 \quad \text{for all } y \in S\bar{x}, \quad (1.1)$$

where  $\langle \cdot, \cdot \rangle$  denote the inner product in  $\mathbb{R}^n$ . If  $h$  is the identity map on  $\mathbb{R}^n$ , IQVI reduces to QVI [4, 5]. When  $C$  is a convex closed subset of  $\mathbb{R}^n$  and for all  $x \in \mathbb{R}^n$ ,  $Sx = C$ , IQVI is nothing but IVI [6, 7]. Aussel et al. [1] stated a main motivation to consider the general IQVI and provided an interesting example [1, Example 1] which confirms that this extension is necessary in a practical sense. They also obtained local/global error bounds for IQVI in terms of standard gap functions

---

Received January 09, 2015; Accepted January 26, 2015.

2010 Mathematics Subject Classification: Primary 49J40, 47J20, 49J50, 52A41.

Key words and phrases: inverse quasi-variational inequalities, gap functions, duality mapping, subdifferential of convex function.

This work was supported by the research grant of the Chungbuk National University in 2014.

such as the residual gap function, the regularized gap function and the D-gap function. However, discussions are made in a finite dimensional Euclidean space. In addition, as pointed out by them, IQVI is still not fully explored.

Motivated by these facts, in this note, we develop an infinite dimensional version of IQVI by investigating some basic properties of the regularized gap function of IQVI in a Banach space. In a Hilbert space setting, extensions of the results in [1] are straightforward by the same argument in [1]. So we pay the attention to Banach spaces.

## 2. Preliminaries

First we define the following IQVI in a Banach space.

**IQVI :** Let  $E$  be a real Banach space and  $E^*$  be its dual space. Given functions  $F : E \rightarrow E^*$ ,  $h : E \rightarrow E$  and a multifunction  $S : E \rightrightarrows E$  with closed convex values  $Sx$  for all  $x \in E$  (for the simplicity of argument,  $Sx$  is assumed to be nonempty), find  $\bar{x} \in E$  such that  $h(\bar{x}) \in S\bar{x}$  and

$$\langle F\bar{x}, y - h(\bar{x}) \rangle \geq 0 \quad \text{for all } y \in S\bar{x}, \quad (2.1)$$

where  $\langle \cdot, \cdot \rangle$  denote the dual pairing on  $E \times E^*$ .

**DEFINITION 2.1.** A function  $g : E \rightarrow \mathbb{R}$  is called a *gap function of IQVI on a subset  $K \subset E$*  if it satisfies

- (i)  $g(x) \geq 0$  for all  $x \in K$ ;
- (ii)  $g(\bar{x}) = 0$ ,  $\bar{x} \in K \Leftrightarrow \bar{x}$  is a solution of IQVI.

As is well-known [2, 4, 1], VI, QVI and IQVI can be equivalently formulated as a minimization problem of certain gap function. This work has a basic concern about the (regularized gap) function  $g_\alpha : E \rightarrow \mathbb{R}$  ( $\alpha > 0$ ) under suitable conditions;

$$g_\alpha(x) := - \inf_{y \in Sx} \left\{ \langle Fx, y - h(x) \rangle + \frac{1}{2\alpha} \|y - h(x)\|^2 \right\}. \quad (2.2)$$

For a proper convex lower semicontinuous function  $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ , the subdifferential of  $f$  at  $x$  is the set

$$\partial f(x) = \{x^* \in E^* \mid \langle x^*, y - x \rangle \leq f(y) - f(x) \text{ for all } y \in E\}.$$

From the definition of subdifferential, we immediately obtain the result below.

PROPOSITION 2.2. *Let  $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex lower semicontinuous function and  $x_0 \in E$ . Define  $g(x) = f(x + x_0)$  for each  $x \in E$ . Then  $\partial g(x) = \partial f(x + x_0)$ .*

In addition, as seen in [9, 2.26 Example, p.27], the following fact is well-known.

PROPOSITION 2.3. *For  $f(x) = \frac{1}{2}\|x\|^2$ , we have*

$$\partial f(x) = J(x) = \{x^* \in E^* \mid \langle x^*, x \rangle = \|x^*\| \|x\| \text{ and } \|x^*\| = \|x\|\}.$$

*$J$  is called the duality mapping for  $E$ .*

In what follows, we need continuity notions of multifunctions below.

DEFINITION 2.4. Let  $X$  and  $Y$  be topological spaces.  $T : X \rightrightarrows Y$  be a multifunction. Then  $T$  is called

- (i) *upper semicontinuous* (u.s.c.) if for each  $x \in X$  and each open set  $V$  in  $Y$  with  $T(x) \subset V$ , there exists an open neighborhood  $U$  of  $x$  in  $X$  such that  $T(y) \subset V$  for each  $y \in U$ ;
- (ii) *lower semicontinuous* (l.s.c.) if for each  $x \in X$  and each open set  $V$  in  $Y$  with  $T(x) \cap V \neq \emptyset$ , there exists an open neighborhood  $U$  of  $x$  in  $X$  such that  $T(y) \cap V \neq \emptyset$  for each  $y \in U$ ;
- (iii) *continuous* if  $T$  is both u.s.c. and l.s.c.

### 3. Main result

THEOREM 3.1. *The function  $g_\alpha$  in (2.2) is a gap function of IQVI on the set  $h^{-1}(S) = \{x \in E \mid h(x) \in Sx\}$ , that is, it satisfies*

- (i)  $g_\alpha(x) \geq 0$  for all  $x \in h^{-1}(S)$ ;
- (ii)  $\bar{x}$  is a solution of IQVI if and only if  $\bar{x} \in h^{-1}(S)$  and  $g_\alpha(\bar{x}) = 0$ .

*Proof.* (i) The result follows from taking  $y = h(x)$  in (2.2) because  $h(x) \in Sx$ .

(ii) ( $\Rightarrow$ ) Let  $\bar{x}$  be a solution of IQVI. Then we have

$$\langle F\bar{x}, y - h(\bar{x}) \rangle \geq 0 \text{ for all } y \in S\bar{x}.$$

Clearly

$$\langle F\bar{x}, y - h(\bar{x}) \rangle + \frac{1}{2\alpha} \|y - h(\bar{x})\|^2 \geq 0 \text{ for all } y \in S\bar{x}.$$

Hence, we get  $g_\alpha(\bar{x}) \leq 0$ . Since  $h(\bar{x}) \in S\bar{x}$  and  $g_\alpha(\bar{x}) \geq 0$  by (i), we see  $g_\alpha(\bar{x}) = 0$ .

( $\Leftarrow$ ) Assume that  $h(\bar{x}) \in S\bar{x}$  and  $g_\alpha(\bar{x}) = 0$ . Then

$$\langle F\bar{x}, y - h(\bar{x}) \rangle + \frac{1}{2\alpha} \|y - h(\bar{x})\|^2 \geq 0 \quad \forall y \in S\bar{x}.$$

Equivalently,

$$\langle F\bar{x}, y \rangle + \frac{1}{2\alpha} \|y - h(\bar{x})\|^2 \geq \langle F\bar{x}, h(\bar{x}) \rangle + \frac{1}{2\alpha} \|h(\bar{x}) - h(\bar{x})\|^2 \quad \forall y \in S\bar{x}.$$

This means that  $h(\bar{x})$  is a solution of the following minimization problem

$$\min_{y \in S\bar{x}} \langle F\bar{x}, y \rangle + \frac{1}{2\alpha} \|y - h(\bar{x})\|^2.$$

By the optimality condition together with Propositions 2.2 and 2.3, we obtain

$$0 \in F\bar{x} + \frac{1}{\alpha} J(h(\bar{x}) - h(\bar{x})) + N_{S\bar{x}}(h(\bar{x}))$$

where  $N_{S\bar{x}}(h(\bar{x})) = \{x^* \in E^* \mid \langle x^*, y - h(\bar{x}) \rangle \leq 0 \text{ for all } y \in S\bar{x}\}$ , the normal cone of  $S\bar{x}$  at  $h(\bar{x})$ . As  $J(0) = \{0\}$ , we get  $-F\bar{x} \in N_{S\bar{x}}(h(\bar{x}))$ , that is,

$$\langle F\bar{x}, y - h(\bar{x}) \rangle \geq 0 \quad \text{for all } y \in S\bar{x},$$

which implies that  $\bar{x}$  is a solution of IQVI. This completes the proof.  $\square$

REMARK 3.2. Theorem 3.1 is an infinite dimensional extension of Aussel et al. [1, Proposition 4.2].

As far as the continuity of the gap function  $g_\alpha$  is concerned, we have the following:

THEOREM 3.3. *Let  $F : E \rightarrow E^*$  ( $E^*$  is endowed with the norm topology),  $h : E \rightarrow E$  and  $S : E \rightrightarrows E$  be continuous. Assume that  $Sx$  be a (nonempty) compact convex set for all  $x \in E$ . Then the gap function  $g_\alpha$  is continuous.*

*Proof.* Note from (2.2) that

$$g_\alpha(x) = \sup_{y \in Sx} \left\{ \langle Fx, h(x) - y \rangle + \frac{1}{2\alpha} \|h(x) - y\|^2 \right\}.$$

Define  $\psi : E \times E \rightarrow \mathbb{R}$  to be  $\psi(x, y) = \langle Fx, h(x) - y \rangle + \frac{1}{2\alpha} \|h(x) - y\|^2$ . As  $E^*$  is equipped with the norm topology,  $\psi$  is clearly continuous. It follows from Berge [2, Theorems 1 and 2, pp.115-116] that

$$g_\alpha(x) = \sup_{y \in Sx} \psi(x, y) = \max_{y \in Sx} \psi(x, y)$$

is continuous.  $\square$

REMARK 3.4. Only for the continuity of  $g_\alpha$ , those of  $F$ ,  $h$  and  $S$  (with the compactness of  $Sx$ ) are sufficient as seen in Theorem 3.3. However, differentiability needs more conditions including differentiability of  $F$  and  $h$  as well as the smoothness of the norm  $\|\cdot\|$ . This can be easily induced by the equation (3.1) below.

From now on, for further discussions of  $g_\alpha$ ,  $E$  is assumed to be a reflexive Banach space. Since the function

$$\psi_\alpha^x(y) = \langle Fx, y - h(x) \rangle + \frac{1}{2\alpha} \|y - h(x)\|^2$$

is strongly convex for each fixed  $x \in E$ , it is coercive (see [8]). Thus  $\psi_\alpha^x$  has a unique minimizer  $z_\alpha(x) \in Sx$  over the closed convex set  $Sx$ . Hence

$$\begin{aligned} g_\alpha(x) &= -\min_{y \in Sx} \psi_\alpha^x(y) = -\psi_\alpha^x(z_\alpha(x)) \\ &= -\langle Fx, z_\alpha(x) - h(x) \rangle - \frac{1}{2\alpha} \|z_\alpha(x) - h(x)\|^2. \end{aligned} \quad (3.1)$$

Then a solution  $\bar{x}$  of IQVI has the following characterization:

THEOREM 3.5.  $\bar{x}$  is a solution of IQVI if and only if  $h(\bar{x}) = z_\alpha(\bar{x})$ .

*Proof.* ( $\Rightarrow$ ) Suppose that  $\bar{x}$  is a solution of IQVI. Since  $z_\alpha(\bar{x})$  is a minimizer of  $\psi_\alpha^{\bar{x}}$  over  $S\bar{x}$ , by the optimality condition together with Propositions 2.2 and 2.3, we have

$$0 \in F\bar{x} + \frac{1}{\alpha} J(z_\alpha(\bar{x}) - h(\bar{x})) + N_{S\bar{x}}(z_\alpha(\bar{x})). \quad (3.2)$$

So, for some  $x^* \in J(z_\alpha(\bar{x}) - h(\bar{x}))$ , we have

$$-F\bar{x} - \frac{1}{\alpha} x^* \in N_{S\bar{x}}(z_\alpha(\bar{x})).$$

Thus,

$$\langle F\bar{x} + \frac{1}{\alpha} x^*, y - z_\alpha(\bar{x}) \rangle \geq 0 \text{ for all } y \in S\bar{x}. \quad (3.3)$$

Taking  $y = h(\bar{x}) \in S\bar{x}$  yields that

$$\begin{aligned} 0 &\geq -\frac{1}{\alpha} \|z_\alpha(\bar{x}) - h(\bar{x})\|^2 = -\frac{1}{\alpha} \langle x^*, z_\alpha(\bar{x}) - h(\bar{x}) \rangle \\ &\geq \langle F\bar{x}, z_\alpha(\bar{x}) - h(\bar{x}) \rangle \geq 0 \end{aligned}$$

because  $\bar{x}$  is a solution of IQVI and  $x^* \in J(z_\alpha(\bar{x}) - h(\bar{x}))$ . This implies that  $h(\bar{x}) = z_\alpha(\bar{x})$ , as desired.

( $\Leftarrow$ ) Assume that  $h(\bar{x}) = z_\alpha(\bar{x}) \in S\bar{x}$ . Since  $J(0) = \{0\}$ , we again obtain from (3.2) and (3.3) that

$$\langle F\bar{x}, y - h(\bar{x}) \rangle \geq 0 \text{ for all } x \in S\bar{x},$$

which means that  $\bar{x}$  is a solution of IQVI. This completes the proof.  $\square$

### References

- [1] D. Aussel, R. Gupta, and A. Mehra *Gap functions and error bounds for inverse quasi-variational inequality problems*, J. Math. Anal. Appl. **407** (2013), 270-280.
- [2] C. Berge, *Topological spaces*, Oliver and Boyd Ltd, London, 1963.
- [3] M. Fukushima, *Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems*, Math. Programming **53** (1992), 99-110.
- [4] F. Giannessi, *Separation of sets and gap functions for quasi-variational inequalities*, in F. Giannessi and A. Maugeri (eds.): *Variational Inequality and Network Equilibrium Problems*, Plenum Press, New York, 1995, 101-121.
- [5] N. Harms, C. Kanzow, and O. Stein, *Smoothness properties of a regularized gap function for quasi-variational inequalities*, Optim. Meth. Software. **29** (2014), 720-750.
- [6] Q. Han and B. S. He, *A predict-correct projection method for monotone variant variational inequalities*, Chin. Sci. Bull. **43** (1998), 1264-1267.
- [7] X. He and H. X. Liu, *Inverse variational inequalities with projection-based solution methods*, Eur. J. Oper. Res. **208** (2011), 12-18.
- [8] S. H. Kum, *A note on a regularized gap function of QVI in Banach spaces*, J. Chungcheong Math. Soc. **27** (2014), 271-276.
- [9] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, 2nd ed., Lecture Notes in Mathematics, Vol. **1364** Springer-Verlag, Berlin/New York, 1993.

\*

Department of Mathematics Education  
 Chungbuk National University  
 Cheongju 362-763, Republic of Korea  
*E-mail*: shkum@cbnu.ac.kr